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The property of linear fractional integro-differential equations on the basis of the collocation method and the operational matrices of cubic B-spline wavelets

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Abstract

In this paper, by using the collocation method the operational matrix of Caputo fractional derivative was constructed for the cubic B-spline scaling function and wavelets. We apply the operational matrix and the Collocation method to solve linear fractional Volterra integro-differential equations. By using the principal characteristic of this technique, we have the operational matrix, which is applied to solve any linear fractional Volterra integro-differential equation by reducing the time, computer memory occupation and convert to a system of linear equations. Finally, the validity and applicability of the new technique will be shown by some numerical examples and convergence analysis.

1. Introduction

There are various applications of Fractional calculus in the fields of engineering and science such as viscoelasticity [1], electrochemistry [2], control [3], Signal Processing [4], porous media [5], diffusion processes [6, 34-42], etc. By developing fractional calculus, the behavior of many systems can be described using the fractional differential and fractional integro-differential systems. Recently, investigations in engineering, science, and other fields have demonstrated that the materials with memory and hereditary effects and dynamical processes including gas diffusion, heat conduction in fractal porous media and sliding mode control can be modeled by fractional order models rather than integer models [7- 9].

In this paper, an operational matrix method of scaling functions and wavelet of cubic B-spline will be suggested to solve the fractional integro-differential equation of the type:

$$ {}_0^C D_x^{\alpha} y(x) = f(x) + p(x)y(x) + \int_0^x K(x,t)y(t)dt \quad \alpha \in \mathbb{R}^+, \quad t \in [0,1], \tag{1}$$

with the initial conditions

$$ y(0) = y_0, y'(0) = y_1, \dots, y^{(\lceil \alpha-1 \rceil)}(0) = y_{\lceil \alpha-1 \rceil}. \tag{2}$$

The method used in this paper is based on using the operational matrix of Caputo fractional derivative for cubic B-spline scaling functions and wavelets with collocation method to reduce the problem under consideration into a system of algebraic equations and solving this system give a numerical solution of the problem. The Caputo definition of fractional derivatives provides initial conditions with clear physical interpretation and that the derivative of a constant is equal to 0. The semi-orthogonal B-spline scaling functions and wavelets and their dual functions used in this paper have compact support, vanishing moments [10, 11]. These properties make many of the operational matrix elements be very small compared with the largest ones. These scaling functions and wavelets can be represented in a closed form such that it is easy to work with them.

In recent years, numerous numerical experiments by wavelets and wavelet operational matrices of fractional integration and derivative are discussed for fractional intgro-differential equations, fractional differential equations and fractional partial differential equations to eliminate the integral and differential operations and reducing the problem into solving a system of algebraic equations.

In [12]-[24] is used from Legendre wavelets, Chebyshev wavelets, B-spline wavelets and CAS wavelets for numerical solution various fractional partial differential equations, fractional Poisson equation, fractional integro-differential equations, fractional population growth model in a closed system, fractional optimal control problems, fractional nonlinear Fredholm integro-differential equations, etc.

Recent works in the numerical solution of linear and nonlinear fractional Volterra integro-differential equations include Adomian decomposition method [26] and [27], Legendre and Chebyshev wavelets method [14] and [21], fractional differential transform method [28], homotopy analysis method [31], etc.

2 Basic definitions

With the development of theories of fractional derivatives and integrals, many definitions emerged, such as Riemann-Liouville and Caputo fractional differential-integral definition.

In this section, it was attempted to present some notations, necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing the results [3].

Definition 2.1 The left Riemann-Liouville fractional integral operator I^{α} of order α (left RLFI) and the left Riemann-Liouville fractional derivative (left RLFD) expressed by:

$$ {}_aI_x^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \quad {}_aD_x^{\alpha}\varphi(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{\varphi(t)}{(x-t)^{\alpha-n+1}}dt. \tag{3}$$

Definition 2.2 The left Caputo fractional derivative (left CFD) is described by:

$$ {}_{C_a}D_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{\varphi^n(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau. \tag{4}$$

Some useful preliminaries of fractional integrals and derivatives operations are given by:

1. The relationship between left RLFD and left CFD is as follows:

$$ {}_{C_a}D_t^{\alpha}\varphi(t) = {}_aD_t^{\alpha}\varphi(t) - \sum_{k=0}^{n-1}\frac{\varphi^k(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}. \tag{5}$$

2. For $n-1<\alpha\leq n$ and $\varphi(t)\in C^{\lceil\alpha\rceil}[0,1]$ the relationship between the Riemann-Liouville integral and Caputo derivative is given by the following expression:

$$ {}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha}\varphi(t) = \varphi(t) - \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{t^j}{j!}\left(\frac{d^j}{dt^j}f\right)(0). \tag{6}$$

($\lceil\alpha\rceil$ referred to the smallest integer greater than or equal to α)

3. As a property for Caputo fractional derivative, we can achieve

$$ {}_{C_0}D_t^{\alpha}(ax-b)_+^n = \begin{cases} 0 & n<\lceil\alpha\rceil \\ a^{\alpha}\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(ax-b)_+^{n-\alpha} & n\geq\lceil\alpha\rceil \end{cases} \tag{7}$$

where in

$$x_+^n = \begin{cases} x^n & x>0 \\ 0 & x\leq 0 \end{cases}$$

for $n\in\mathbb{N}\cup\{0\}$.

3 Scaling functions and wavelets of cubic B-spline

This section is a brief explanation about the multi-resolution analysis (MRA) of the B-spline scaling functions and wavelet. It also expresses some of their properties, as well as some of the concepts, are used in the following sections [10, 29, 30].

A scaling function on the real line is a unique function $\varphi \in L^2(\mathbb{R})$ as the collection of its integer translations product subspace of V_0. That's mean, $V_0 = \overline{span}\{\varphi(x-k), k \in \mathbb{Z}\}$. Now consider, $V_j = \overline{span}\{\varphi(2^j x - k), k \in \mathbb{Z}\}$.
Therefore, $\{V_j\}_{j\in\mathbb{Z}}$ makes MRA on the $L^2(\mathbb{R})$.

Definition 3.1 A wavelet is a function as $\psi \in L^2(\mathbb{R})$ such that $\{\psi(2^j x - k), k, j \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$. The wavelets, especially B-spline wavelets have a particularly desirable property such as compact support and vanishing moment of order m. According to the terms of the MRA and the orthogonal decomposition theorem can be achieved the closed subspaces of W_j that is orthogonal complement V_j in V_{j+1} such that we have:

$$L^2(\mathbb{R}) = \sum_{j\in\mathbb{Z}} \oplus W_j = V_{j_0} \oplus \sum_{j=j_0}^{\infty} \oplus W_j = \overline{\bigcup_{j\in\mathbb{Z}} V_j}. \tag{8}$$

Regarding the concepts mentioned, it can be concluded that to find a Riesz basis for $L^2(\mathbb{R})$, it is enough to obtain a Riesz basis for every W_j. Therefore, the problem to be converted in this way ψ function find out such that $\{\psi(2^j x - k), k \in Z\}$ is a Riesz basis for W_j. The wavelet of ψ is constructed with MRA and scaling functions such that the collection of its integer translations on the real line can product subspace of W_0. With this assumption can attain a Riesz basis for W_j also Riesz basis of $\{\psi(2^j x - k), k, j \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$.

Definition 3.2 The scaling function of cubic B-spline (Fourth-order B-spline) can be expressed as follows:

$$B_4(x) = \frac{1}{6}\sum_{k=0}^{4} \binom{4}{k} (-1)^k (x-k)_+^3. \tag{9}$$

Then, corresponding with scaling function, the cubic B-spline wavelet ψ is given by[29]:

$$\psi_4(x) = \sum_{k=0}^{10} \frac{(-1)^k}{8} \sum_{l=0}^{4} \binom{4}{k} B_8(k-l+1) B_4(2x-k) = \sum_{k=0}^{10} \frac{(-1)^k}{48} \sum_{l=0}^{4}\sum_{t=0}^{4} \binom{4}{k}\binom{4}{t} (-1)^t B_8(k-l+1)(2x-k)_+^k, \tag{10}$$

where in $B_m(k) = \frac{k}{m-1} B_{m-1}(k) + \frac{m-k}{m-1} B_{m-1}(k-1)$ and $B_1(k) = \chi[0,1](k)$ is the characteristic function of [0, 1].

We put $\varphi_{j,k} = B_4(2^j x - k)\chi[0,1](x)$, $\psi_{j,k} = \psi_4(2^j x - k)\chi[0,1](x)$ hence $V_j = \overline{span}\{\varphi_{j,k}, k \in \mathbb{Z}\}$ and $W_j = \overline{span}\{\psi_{j,k}, k \in \mathbb{Z}\}$. The support of cubic B-spline scaling function and wavelet is [0, 4] and [0, 7] Respectively. Then
support $(\varphi_{j,k}) = [2^{-j}k, 2^{-j}(k+4)] \cap [0,1]$.
Define the set of indices

$$Q_j = \{k: [2^{-j}k, 2^{-j}(k+4)] \cap [0,1] \neq \varnothing\}, \quad j \in \mathbb{Z}.$$

It is obvious that $\min Q_j = -3$ and $\max Q_j = 2^j - 1, j \in \mathbb{Z}$.

Then $\varphi_{j,k}$ for $k = -3, -2, -1$ are left boundary, for $k = 2^j - 3, \ldots, 2^j - 1$ are right boundary and for $k = 0, \ldots, 2^j - 4$ are inner scaling function. Hence, according to the two-scale relation for the fourth-order B-spline wavelet, we have left, inner and right for $k = -3, \ldots, 2^j - 4$. The support of the cubic B-spline wavelet occupies 7 portions. Now, if [0,1] apportion to 2^j sub-interval which has 2^{-j} length. Therefore, the lowest level to have at least cubic wavelet B-spline, must be satisfied: $2^j \geq 7$ then $j_0 = j = 3$ is the lowest level.

The B-spline wavelet is semi-orthogonal, i.e.

$$< \psi_{j,k}(x), \psi_{i,k}(x) >= 0 \quad i \neq j$$

Hence, for the better use of wavelets in approximation theory is utilized Dual Multi Resolution Analysis (DMRA) is produced by subspaces of $\tilde{V}_j = \{\tilde{\varphi}_{j,k}, k \in \mathbb{Z}\}$ and $\widetilde{W}_j = \{\tilde{\psi}_{j,k}, k \in \mathbb{Z}\}$ such that

$$\tilde{V}_j \perp W_j\,, \quad V_j \perp \widetilde{W}_j \quad \text{and} \quad \widetilde{W}_j \perp W_j' \quad \text{for } j \neq j'.$$

3.1 The functions approximation by subspaces of V_{j_0} and W_j $(j \geq j_0, j_0 \in \mathbb{Z})$

With the above assumptions and according to (8), each function $f(x) \in L^2(\mathbb{R})$ over [0,1] can be expanded by[22, 29]:

$$f(x) \simeq \sum_{k=-3}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=-3}^{2^j-4} d_{j,k}\psi_{j,k}(x).$$

If the orthogonal projection of $f(x)$ is considered in the subspaces and be truncated in definite j_u, the above relation will be rewritten as follows:

$$f(x) \simeq \sum_{k=-3}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{j_u}\sum_{k=-3}^{2^j-4} d_{j,k}\psi_{j,k}(x). \tag{11}$$

The coefficients of $c_{j_0,k}$ and $d_{j,k}$ were obtained from the following equation:

$$c_{j_0,k} = \int_0^1 f(x)\tilde{\varphi}_{j,k}(x), \qquad k = -3\ldots 2^{j_0} - 1, \qquad d_{j_0,k} = \int_0^1 f(x)\tilde{\psi}_{j,k}(x), k = -3,\ldots,2^j - 4, j = j_0,\ldots,j_u$$

$\tilde{\varphi}_{j,k}(x)$ and $\tilde{\psi}_{j,k}(x)$ can be obtained by linear combination from $\varphi_{j,k}(x)$ and $\psi_{j,k}(x)$ functions.

We put

$$\begin{aligned}\gamma_{j_u}(x) &= [\varphi_{j_0,-3}(x),\ldots,\varphi_{j_0,2^{j_0}-1}(x),\psi_{j_0,-3}(x),\ldots,\psi_{j_0,2^{j_0}-4}(x),\psi_{j_0+1,-3}(x),\ldots,\psi_{j_u,2^{j_u}-4}(x)]^T \\ \kappa &= [c_{j_0,-3},\ldots,c_{j_0,2_0^j-1},d_{j_0,-3},\ldots,d_{j_0,2^{j_0}-4},d_{j_0+1,-3},\ldots,d_{j_u,2^{j_u}-4}]^T,\end{aligned} \tag{12}$$

with $j_0 = 3$, (12) is a $(2^{j_u+1} + 3) \times 1$ matrix, therefore (11) in the form of a matrix can be expressed as follows:

$$f(x) = \kappa^T\gamma_{j_u}(x). \tag{13}$$

If the following interpolation points are selected:

$$x_1 = 0, x_2 = \frac{1}{2^{j_u+1}+1},\ x_k = \frac{k-2}{2^{j_u+1}+1},\ 3 \leq k \leq 2^{j_u+1} + 1,\ x_{2^{j_u+1}+2} = 1 - \frac{1}{2^{j_u+1}+1},\ x_{2^{j_u+1}+3} = 1. \tag{14}$$

(13) is given by:

$$\hat{f} = \kappa^T B_{j_u},\ \hat{f} = [f(x_1), f(x_2),\ldots, f(x_{2^{j_u+1}+3})], B_{j_u} = [\gamma_{j_u}(x_1),\gamma_{j_u}(x_2),\ldots,\gamma_{j_u}(x_{2^{j_u+1}+3})]. \tag{15}$$

By dissolving (15) system will have:

$$\kappa^T = \hat{f}B_{j_u}^{-1}. \tag{16}$$

4 The operational matrix of Caputo fractional derivative for cubic B-spline scaling functions and wavelets

To simplify the calculations of differential fractional integral equations is obtained Caputo fractional derivative of the cubic B-spline scaling function and wavelets afterwards operating matrix can be built in a way that will be mentioned. This operational matrix converts the linear fractional differential-integral equation to a linear algebraic system such that the approximate answer is obtained easily.

Theorem 1 *The Caputo fractional derivative of the cubic B-spline scaling functions is obtained from the following equation.*

$${}_0^C D_x^\alpha \varphi_{j_0,k}(x) = (\frac{2^{j_0\alpha}}{\Gamma(4-\alpha)}\sum_{i=0}^{4}\binom{4}{i}(-1)^i(2^{j_0}x - k - i)_+^{3-\alpha})\chi_{[0,1]}(x).$$

Proof.

$$ {}_0^C D_x^\alpha \varphi_{j_0,k}(x) = {}_0^C D_x^\alpha B_4(2_0^j x - k)\chi_{[0,1]}(x) = (\frac{1}{6}\sum_{i=0}^{4}\binom{4}{i}(-1)^i {}_0^C D_x^\alpha (2^{j_0}x - i - k)_+^3)\chi_{[0,1]}(x) $$

$$ = \frac{2^{j_0\alpha}}{\Gamma(4-\alpha)}(\sum_{i=0}^{4}\binom{4}{i}(-1)^i {}_0^C D_x^\alpha (2^{j_0}x - i - k)_+^{3-\alpha})\chi_{[0,1]}(x). $$

Lemma 2 *The Caputo fractional derivative of the cubic B-spline wavelets is obtained from the following equation.*

$$ {}_0^C D_x^\alpha \psi_{j,k}(x) = (\frac{2^{(j+1)\alpha}}{\Gamma(4-\alpha)}\sum_{r=0}^{10}\frac{(-1)^r}{8}\sum_{l=0}^{4}\sum_{t=0}^{4}\binom{4}{l}\binom{4}{t}(-1)^t B_8(r-l+1)(2^{j+1}x - t - r - 2k)_+^{3-\alpha})\chi_{[0,1]}(x) $$

Proof. Similarly to the previous theorem achieved by replacing the (10) in (7).

We put:

$$ {}_0^C D_x^\alpha \varphi_{j_0,k}(x) = \eta_{j_0,k}(x), k = -3, \dots, 2^{j_0-1}, \quad {}_0^C D_x^\alpha \psi_{j,k}(x) = \beta_{j,k}, k = -3, \dots, 2^j - 4, j = j_0, \dots j_u \quad (17) $$

Therefore

$$ {}_0^C D_x^\alpha \gamma_{j,u}(x) = [\eta_{j_0,-3}(x), \dots \text{m}\ \eta_{j_0,2^{j_0-1}}(x), \beta_{j_0,-3}, \dots, \beta_{j_u,2^j-4} = W_{j_u}(x). \quad (18) $$

4.1 The operational matrix of the nth order derivative

To obtain α order derivative ($\alpha > 1$) need to the derivative of integer order, then derivative of integer order is expressed in the following way:

If $\tilde{\gamma}_{j_u}(x)$ be considered the dual matrix of the cubic B-spline scaling functions and wavelets, $\tilde{\gamma}_{j_u}(x) = T\gamma_{j_u}(x)$.

Parties were multiplied in $\gamma_{j_u}^T(x)$ and then integrated; therefore, the following can be obtained:

$I = T\int_0^1 \gamma_{j_u}(x)\gamma_{j_u}^T(x)dx$ then $T^{-1} = \int_0^1 \gamma_{j_u}(x)\gamma_{j_u}^T(x)dx$.

By putting $P = T^{-1}$ we have $\tilde{\gamma}_{j_u}(x) = P^{-1}\gamma_{j_u}(x)$.

Now, derivative of $\gamma_{j_u}(x)$ vector function can be considered as follows:

$$ \gamma_{j_u}'(x) = D_{j_u}\gamma_{j_u}(x), $$

that D_{j_u} is derivative of the operational matrix corresponding to the cubic B-spline scaling functions and wavelets over $[0,1]$ is calculated as follows:

$$ D_{j_u} = \int_0^1 \gamma_{j_u}'(x)\tilde{\gamma}_{j_u}^T(x)dx = \int_0^1 \gamma_{j_u}'(x)(P^{-1}\gamma_{j_u}(x))^T dx = E_\gamma (P^{-1})^T, \quad (19) $$

where E_γ is square matrix of order $2^{j_u+1} + 3$. According to (19) equation, nth order derivative of $\gamma_{j_u}(x)$ vector is given by:

$$ \gamma_{j_u}^{(n)}(x) = D_{Ju}^n \gamma_{j_u}(x) = (E_\gamma (P^{-1})^T)^n \gamma_{j_u}(x). \quad (20) $$

4.2 The operational matrix of the fractional order derivative

In this section, at first, the operational matrix of fractional derivative from α order, $0 < \alpha \leq 1$ is obtained for $\gamma_{j_u}(x)$. Then, using this and the operational matrix of integer order derivative, the operational matrix of α order Caputo fractional derivative $m < \alpha \leq m+1, (m \in \mathbb{N})$ will be calculated.

If $0 < \alpha \leq 1$, then Caputo fractional derivative of the cubic B-spline scaling function and wavelet can be given by:

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = W_{j_u}(x) = D_\alpha\, \gamma_{j_u}(x). \quad (21) $$

Using relations (13), (15) and (16), the operational matrix Caputo fractional derivative can be obtained as follows:

$$D_\alpha = G_{j_u} B_{j_u}^{-1},\ where\ \ G_{j_u} = [W_{j_u}(x_1), W_{j_u}(x_2), \dots, W_{j_u}(x_{2^{j_u+1}+3})]. \tag{22}$$

If $\alpha > 1$, the operational matrix of derivative of order α for the cubic B-spline scaling function and wavelet can be expressed in the form of the following theorem:

Theorem 3 *for any $n < \alpha \leq n+1$ $(n \in \mathbb{N})$, the operational matrix of the fractional order derivative is given by:*

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \left(\widehat{D}_\alpha - \sum_{k=1}^{n} D_{j_u}^\alpha \gamma_{j_u}(0) H B_{j_u}^{-1} \right) \gamma_{j_u} = \Delta_\alpha \gamma_{j_u}(x)$$

$$\text{where } \widehat{D}_\alpha = D_{\alpha\text{-}u}\ \ D_{j_u}^\alpha, H = \frac{1}{\Gamma(k\text{-}\alpha+1)} \left(x_1^{k\text{-}\alpha}, x_2^{k\text{-}\alpha}, \dots, x_{2^{j_u+1}+3}^{k\text{-}\alpha} \right). \tag{23}$$

Proof. Using left CFD(4) definition, we have

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \frac{1}{\Gamma(n+1-\alpha)} \int_0^x \frac{\gamma_{j_u}^{n+1}(x)(\tau)}{(t-\tau)^{\alpha-n}} d\tau.$$

Applying the relationship between left RLFD and left CFD (5) is obtained:

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \frac{1}{\Gamma(n+1-\alpha)} \frac{d^{n+1}}{dx^{n+1}} \int_0^x \frac{\gamma_{j_u}(\tau)}{(x-\tau)^{\alpha-n}} d\tau - \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}.$$

We put $\alpha = \beta + n, \beta \in (0,1]$ and $n \in \mathbb{N}$; then:

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \frac{1}{\Gamma(1-\beta)} \frac{d^{n+1}}{dx^{n+1}} \int_0^x \frac{\gamma_{j_u}(\tau)}{(x-\tau)^{\beta}} d\tau - \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}$$

$$= \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_0^x \frac{\gamma_{j_u}(\tau)}{(x-\tau)^{\beta}} d\tau \right) - \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}.$$

According to the relationship between left RLFD and left CFD (5) for $\beta \in (0,1]$ will have:

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(1-\beta)} \int_0^x \frac{\gamma_{j_u}'(\tau)}{(x-\tau)^{\beta}} d\tau + \frac{\gamma_{j_u}(0) x^{-\beta}}{\Gamma(1-\beta)} \right) - \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}.$$

Therefore, using left CFD, (21) and (20), we have

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \frac{d^n}{dx^n} \left(D_\beta \gamma_{j_u}(x) + \frac{\gamma_{j_u}(0) x^{-\beta}}{\Gamma(1-\beta)} \right) - \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha} = D_\beta D_{j_u}^n \gamma_{j_u}(x) + \frac{\gamma_{j_u}(0) x^{-\beta-n}}{\Gamma(1-n-\beta)} - $$
$$ \sum_{k=0}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha} = D_\beta D_{j_u}^n \gamma_{j_u}(x) - \sum_{k=1}^{n} \frac{\gamma_{j_u}^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha} = \widehat{D}_\alpha \gamma_{j_u}(x) - \sum_{k=1}^{n} \frac{D_{j_u}^k \gamma_{j_u}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}.$$

According to (15) and (16), the following is obtained:

$$ {}_0^C D_x^\alpha \gamma_{j_u}(x) = \left(\widehat{D}_\alpha - \sum_{k=1}^{n} D_{j_u}^k \gamma_{j_u}(0) H B_{j_u}^{-1} \right) \gamma_{j_u}(x) = \Delta_\alpha \gamma_{j_u}(x).$$

5 The Linear Fractional Volterra Integro-Differential Equations

In this section, the linear fractional Volterra integro-differential equations are solved by using the operational matrix that was created in the previous section. Then, with the collocation method, we convert them into a system of linear equations. Suppose $n < \alpha \leq n+1$ $(n \in \mathbb{N} \cup \{0\})$, consider linear fractional Volterra integro-differential equation with initial conditions:

$$ {}_0^C D_x^\alpha y(x) = p(x) y(x) + f(x) + \int_0^x K(x,t) y(t) dt,\ \ y(0) = y_0, y'(0) = y_1, \dots, y^{(\lceil \alpha-1 \rceil)}(0) = y_{\lceil \alpha-1 \rceil}. \tag{24}$$

Without the loss of generality, we can assume $0 \leq x \leq 1$, also the functions of $f, p: [0,1] \to \mathbb{R}$ and $K: [0,1]— [0,1] \to \mathbb{R}$ are given and supposed to be sufficiently smooth. Now, for solving equation (24), the $y(x)$ function is approximated by $\gamma_{j_u}(x)$ as follows:

$$y(x) = S^T \gamma_{j_u}(x). \tag{25}$$

Using equations (21), (22) and (23), the fractional derivative of unknown function is expanded by:

$${}_0^C D_x^\alpha y(x) = \begin{cases} S^T \mathrm{D}_\alpha \gamma_{j_u}(x), & 0 < \alpha \leq 1 \\ S^T \Delta_\alpha \gamma_{j_u}(x) & n < \alpha \leq n+1, n \in \mathbb{N} \end{cases} \tag{26}$$

We have

$$g(x) = \int_0^x k(x,y) y(t) dt = K^T \gamma_{j_u}(x), \tag{27}$$

where $K^T = \hat{A} B_{j_u}^{-1}$ and $\hat{A} = [g(x_1), g(x_2), \ldots, g(x_{2^{j_u+1}+3})]$. By substituting (26) and (27) in equation (24), the following is obtained:

$$S^T D_\alpha \gamma_{j_u}(x) = S^T p(x) \gamma_{j_u}(x) + f(x) + S^T K^T \gamma_{j_u}(x), \ \ 0 < \alpha \leq 1. \tag{28}$$

$$S^T \Delta_\alpha \gamma_{j_u}(x) = S^T p(x) \gamma_{j_u}(x) + f(x) + S^T K^T \gamma_{j_u}(x), \ \ n < \alpha \leq n+1, \ \ n \in \mathbb{N}. \tag{29}$$

By considering $\lambda(x)$ as follows:

$$\lambda(x) = D_\alpha - p(x) I_{2^{j_u+1}+3} - K^T, \ 0 < \alpha \leq 1.$$

$$\lambda(x) = \Delta_\alpha - p(x) I_{2^{j_u+1}+3} - K^T, \ \ n < \alpha \leq n+1, n \in \mathbb{N}.$$

(28), (29) is converted to the following nonlinear system

$$S^T \lambda(x) \gamma_{j_u}(x) = f(x). \tag{30}$$

By using (20), (25) in the initial condition is obtained

$$S^T D_{j_u}^i \gamma_{j_u}(0) = y_i, \quad i = 0, \ldots, n. \tag{31}$$

To solve by this collocation method, the collocation points of $x_i, i = n+2, \ldots, 2^{j_u+1} + 3$ were defined in (14) and put in (30), hence these equations and (31) constitute a linear system of equations with $2^{j_u+1} + 3$ unknowns and equations. By dissolving it the variables of $s_i, \ i = 1, \ldots, 2^{j_u+1} + 3$ obtain which can be employed to find the function of $y(x)$.

6 Convergence analysis

At the beginning of this section, the error upper bound of approximation answer is investigated by using the cubic B-spline scaling functions and wavelets. Then, the convergence of this method is shown.

Theorem 4 *For the m-th order B-spline wavelet if y_{j_u} is the function approximation of y that the error upper bound of approximation is given by*

$$\| y - y_{j_u} \| \leq C_m 2^{-j_u m} \| y^{(m)} \|_2 , C_m = \sqrt{\frac{B_{2m}}{(2m)!}}, \tag{32}$$

where B_{2m} is a Bernoulli number of order $2m$ [32].

The following theorem illustrates the error upper bound of $y_{j_u}(x)$ as an approximation of $y(x)$ for linear fractional Volterra integro-differential equation which is derived from the method presented in this article; also, it shows this method is convergence.

Theorem 5 *Suppose that $M_1 = max_{x \in [0,1]} |p(x)|$, $M_2 = max_{(x,t) \in [0,1] \times [0,1]} |k(x,t)|$ and $\tilde{y}_{j_u}(x)$ is an approximation of $y(x)$ for equation (24) which is derived from the method presented in this article then the error upper bound for $y(x)$ that applies in equation (24) can be obtained from the following relationship.*

$$|y(x) - \tilde{y}_{j_u}(x)| \leq C_m 2^{-j_u m} \| y^{(m)} \|_2 \left(\frac{M_1}{\Gamma(\alpha+1)} + \frac{M_2}{\Gamma(\alpha+2)}\right), \tag{33}$$

where $\lim_{j_u \to \infty} \tilde{y}_{j_u}(x) = y(x)$.

Proof. By applying ${}_a I_x^\alpha$ integral operation on both sides (24) we obtain:

$$y(x) = c + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)y(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt + \frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)y(t)dt\,, \quad (34)$$

$\tilde{y}_{j_u}(x)$ approximation answer satisfies the following relationship

$$\tilde{y}_{j_u}(x) = c + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)\tilde{y}_{j_u}(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt + \frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)\tilde{y}_{j_u}(t)dt. \quad (35)$$

Subtracting relation (35) from relation (34), we obtain:

$$y(x) - \tilde{y}_{j_u}(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)(y(x) - \tilde{y}_{j_u}(t))dt + \frac{1}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}k(x,t)(y(x) - \tilde{y}_{j_u}(t))dt.$$

Considering that $x \in [0,1]$ and the assumption of theorem, we can write :

$$|y(x) - \tilde{y}_{j_u}(x)| \le \frac{1}{\Gamma(\alpha)}C_m 2^{-j_u m}M_1 \parallel y^{(m)} \parallel_2 \int_0^x (x-t)^{\alpha-1}dt + \frac{1}{\Gamma(\alpha+1)}C_m 2^{-j_u m}M_2 \parallel y^{(m)} \parallel_2 \int_0^x (x-t)^{\alpha}dt$$

$$\le \frac{1}{\Gamma(\alpha)}C_m 2^{-j_u m}M_1 \parallel y^{(m)} \parallel_2 \frac{1}{\alpha} + \frac{1}{\Gamma(\alpha+1)}C_m 2^{-j_u m}M_2 \parallel y^{(m)} \parallel_2 \frac{1}{\alpha+1} = C_m 2^{-j_u m} \parallel y^{(m)} \parallel_2 \left(\frac{M_1}{\Gamma(\alpha+1)} + \frac{M_2}{\Gamma(\alpha+2)}\right).$$

and it is clear if $j_u \to \infty$ then $\tilde{y}_{j_u}(x) \to y(x)$.

Table 1: The results for example by cubic B-spline scaling functions and wavelets($j_u = 5$).

t_i	$y_{j_u=5}$	the absolute error[33]	the absolute error of the proposed method
0.1	0.00100000005818298449	1.688×10^{-8}	5.8×10^{-11}
0.5	0.124999999686091046	2.1102×10^{-6}	3.0×10^{-10}
1.0	0.99999997354647996	1.68816×10^{-5}	2.65×10^{-8}

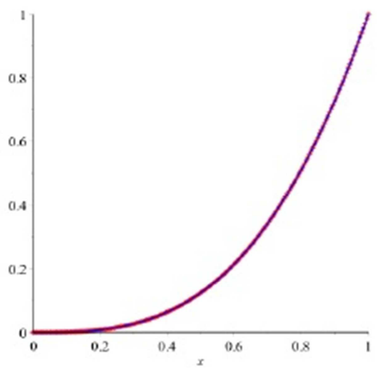
7 Numerical example

In this section, we apply the presented method in this paper to solve three examples for the validity and applicability of the new technique.

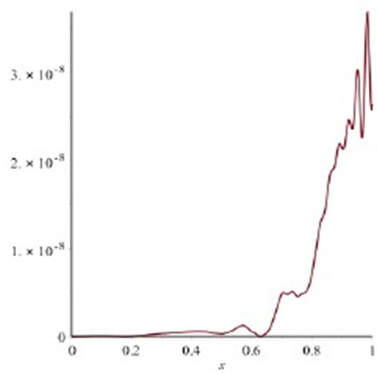
Example 1. Consider the linear fractional Volterra integro-differential equation

$$ {}_0^C D_t^{\frac{3}{4}} y(t) = \left(-\frac{t^2 e^t}{5}\right)y(t) + \frac{6\sqrt[4]{t^9}}{\Gamma(3.25)} + \int_0^t e^t s y(s)ds,$$

with initial condition $y(0) = 0$. The exact solution of this problem is $y(t) = t^3$, This problem is solved by the proposed method for $j_u = 5$. Figure 1 shows the exact and approximate solutions and the absolute error of Example. This problem has also been solved with the collocation method and B-spline functions in [33]. Table 1 demonstrates the approximate answer and absolute error of the proposed method and the method presented in [33] in some arbitrary points.



(a) Approximation and exact solution



(b) Absolute error

Figure 1: figures of example 1

Example 2. Consider the fractional integro-differential equation

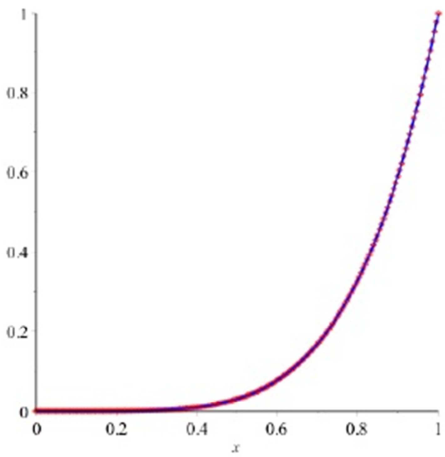
$$C_0 D_t^{\frac{1}{2}} y(t) = (\cos(t) - \sin(t))y(t) + f(t) + \int_0^t \sin(s) t y(s) ds,$$

$$f(t) = \frac{120 t^{\frac{9}{2}}}{\Gamma(5.5)} + t^6 \cos(t) - 4t^5 \sin(t) - t^5 \cos(t) - 20t^4 \cos(t) + 60t^3 \sin(t) + 120t^2 \cos(t) - 120t\sin(t),$$

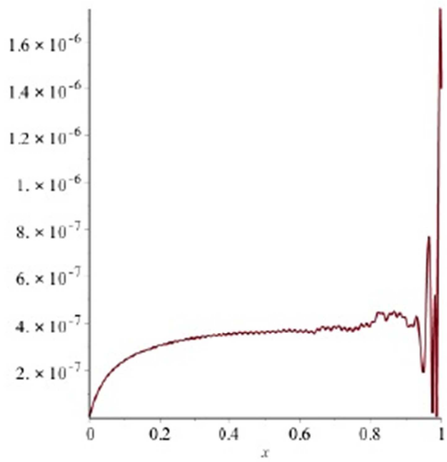
with initial condition $y(0) = 0$ and exact solution $y(t) = t^5$. This problem is solved by the proposed method for $j_u = 5$. Table 2 and Figure 2 show the exact and approximate solutions and absolute error of Example 2 in some arbitrary points.

Table 2: The results for example 2 by cubic B-spline scaling functions and wavelets($j_u = 5$).

t_i	$y_{j_u=5}$	Absolute error
0.0	$2.18546441755966650 \times 10^{-9}$	2.185×10^{-9}
0.1	0.0000097589273767107471	2.410×10^{-7}
0.2	0.000319695889343328305	3.041×10^{-7}
0.3	0.00242966451124808878	3.354×10^{-7}
0.4	0.0102396434188530561	3.565×10^{-7}
0.5	0.0312496437372003014	3.562×10^{-7}
0.6	0.0777596297137816833	3.702×10^{-7}
0.7	0.168069616380019993	3.836×10^{-7}
0.8	0.327679607279856899	3.927×10^{-7}
0.9	0.590489582065617968	4.179×10^{-7}
1.0	0.99999860044204192	1.39×10^{-6}



(a) Approximation and exact solution



(b) Absolute error

Figure 2: figures of example 2

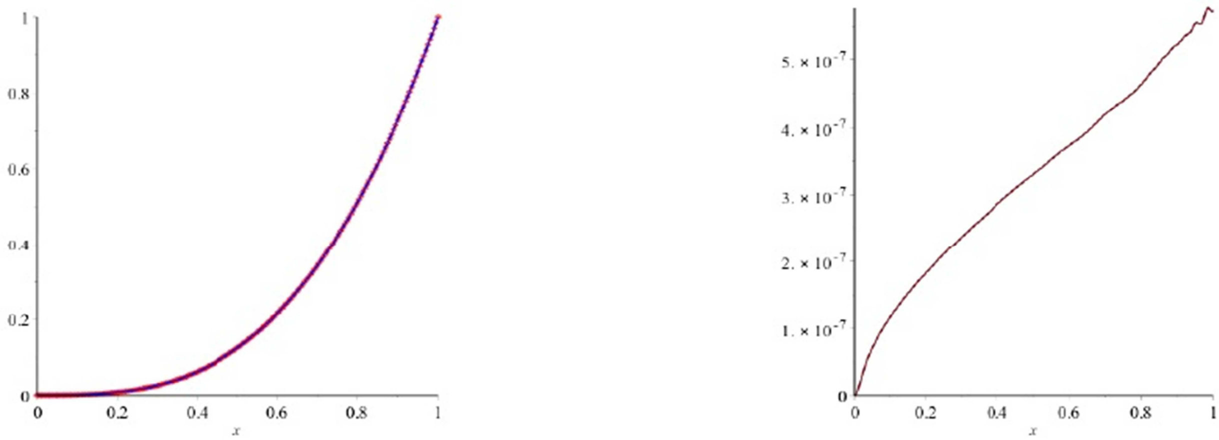
Example 3. Consider the fractional integro-differential equation

$$C_0 D_t^{\frac{7}{4}} y(t) = \frac{6t^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} + (-\frac{t^2 e^t}{5}) y(t) + \int_0^t e^t s y(s) ds,$$

with initial conditions $y(0) = 0, y'(0) = 0$ and exact solution $y(t) = t^3$. This problem is solved by the proposed method for $j_u = 5$. Table 3 and Figure 3 show the exact and approximate solutions and absolute error of Example 3 in some arbitrary points.

Table 3: The results for example 3 by cubic B-spline scaling functions and wavelets($j_u = 5$).

t_i	$y_{j_u=5}$	Absolute error
0.0	$2.29423441755966650 \times 10^{-17}$	2.294×10^{-17}
0.1	0.0000097589273767107471	1.173×10^{-7}
0.2	0.000319695889343328305	1.838×10^{-7}
0.3	0.00242966451124808878	2.381×10^{-7}
0.4	0.0102396434188530561	2.863×10^{-7}
0.5	0.0312496437372003014	3.306×10^{-7}
0.6	0.0777596297137816833	3.734×10^{-7}
0.7	0.168069616380019993	4.202×10^{-7}
0.8	0.327679607279856899	4.640×10^{-7}
0.9	0.590489582065617968	5.233×10^{-7}
1.0	0.99999860044204192	5.737×10^{-7}



(a) Approximation and exact solution (b) Absolute error

Figure 3: figures of example 3

8 Conclusion

In the present work, the collocation method and the scaling functions and wavelets of the cubic B-spline have been successfully used for the operational matrix of Caputo fractional derivative and then the linear fractional Volterra integro-differential equation has been reduced to solve a system of algebraic equations. The operational matrix can be used for solving different problems without doing the new calculations. Numerical results show that the approximation solution fairly matches with the exact solution. The upper bound of error exponentially decreases by growing of approximation space.

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